On rationalizability in weighted maxmin games

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1. Rationalizable strategies in a vector-valued game $G = \{(A^i, u^i)_{i \in N}\}$

- Set of players: $N = \{1, ..., n\}$.
- Sets of strategies: A^i. $A = \times_{i \in N} A^i$.
- Strategy for player i: $a^i \in A^i$. Strategy profile: $a \in A$.
- Vector-valued utility function for player i: $u^i : A \to \mathbb{R}_+^{s^i}$, $u^i := (u_1^i, \ldots, u_{s^i}^i)$, $J^i = \{1, ..., s^i\}$

$a^i \in \mathcal{B}_{X^i}(a^{-i})$, with $X^i \subseteq A^i$, if there is no $\hat{a}^i \in X^i$ such that $u^i(\hat{a}^i, a^{-i}) \geq u^i(a^i, a^{-i})$.
$a^i \in \tilde{\mathcal{B}}_{X^i}(a^{-i})$, if there is no $\hat{a}^i \in X^i$ such that $u^i(\hat{a}^i, a^{-i}) > u^i(a^i, a^{-i})$.

Definition 1.

a) a^i is rationalizable if $a^i \in \mathcal{R}^i(G) = \cap_{k=0}^{\infty} \mathcal{R}_k^i(G)$, where $\mathcal{R}_0^i(G) = A^i$ and for all $k \geq 1$, $\mathcal{R}_k^i(G) = \{a^i \in \mathcal{R}_{k-1}^i(G)$: there exists $a^{-i} \in \times_{j \neq i} \mathcal{R}_{k-1}^j(G)$ such that $a^i \in \mathcal{B}_{\mathcal{R}_{k-1}^i}(a^{-i})\}$.

b) a^i is weak rationalizable if $a^i \in \tilde{\mathcal{R}}^i(G) = \cap_{k=0}^{\infty} \tilde{\mathcal{R}}_k^i(G)$, where $\tilde{\mathcal{R}}_0^i(G) = A^i$ and for all $k \geq 1$, $\tilde{\mathcal{R}}_k^i(G) = \{a^i \in \tilde{\mathcal{R}}_{k-1}^i(G)$: there exists $a^{-i} \in \times_{j \neq i} \tilde{\mathcal{R}}_{k-1}^j(G)$ such that $a^i \in \tilde{\mathcal{B}}_{\tilde{\mathcal{R}}_{k-1}^i}(a^{-i})\}$.

$a \in A$ is rationalizable (weak rationalizable) if for all $i \in N$, $a^i \in \mathcal{R}^i(G)$ ($a^i \in \tilde{\mathcal{R}}^i(G)$).

$\mathcal{R}(G)$ ($\tilde{\mathcal{R}}(G)$) set of rationalizable (weak rationalizable) strategy profiles for G.

Lemma 1. a) For $a^{-i} \in \times_{j \neq i} \mathcal{R}_k^j(G)$, if $a^i \in \mathcal{B}_{\mathcal{R}_k^i}(a^{-i})$ then $a^i \in \mathcal{B}_{A^i}(a^{-i})$.
b) For $a^{-i} \in \times_{j \neq i} \tilde{\mathcal{R}}_k^j(G)$, if $a^i \in \tilde{\mathcal{B}}_{\tilde{\mathcal{R}}_k^i}(a^{-i})$ then $a^i \in \tilde{\mathcal{B}}_{A^i}(a^{-i})$.
$\mathcal{R}_k^i(G) = \{a^i \in \mathcal{R}_{k-1}^i(G)$: there exists $a^{-i} \in \times_{j \neq i} \mathcal{R}_{k-1}^j(G) \; / a^i \in \mathcal{B}_{A^i}(a^{-i})\}$.

2. Rationalizable strategies in a weighted maxmin game $G^\gamma = \{(A^i, w_{\gamma^i}^i)_{i \in N}\}$

- Value function for the strategy profile $a \in A$, $\quad w_{\gamma^i}^i(a) = \min_{j \in J^i} \left\{ \frac{u_j^i(a)}{\gamma_j^i} \right\}$.
- γ_j^i: parameter of importance that player i assigns to the j-th component of her utility function. If $\gamma_j^i = 0$ for some $j \in J^i$, then $\frac{u_j^i(a)}{\gamma_j^i}$ is not computed. $\gamma^i = (\gamma_j^i)_{j \in J^i}$, $\gamma = (\gamma^i)_{i \in N}$.

Let $R_0^i(G^\gamma) = A^i$ and for $k \geq 1$, $R_k^i(G^\gamma) = \{a^i \in R_{k-1}^i(G^\gamma)$: there exists $a^{-i} \in \times_{j \neq i} R_{k-1}^j(G^\gamma)$ with $w_{\gamma^i}^i(a^i, a^{-i}) \geq w_{\gamma^i}^i(\hat{a}^i, a^{-i})$ for all $\hat{a}^i \in R_{k-1}^i(G^\gamma)\}$.
$a^i \in A^i$ is rationalizable in G^γ if $a^i \in R^i(G^\gamma)$ where $R^i(G^\gamma) = \cap_{k=0}^{\infty} R_k^i(G^\gamma)$.

$R(G^\gamma) = \times_{i \in N} R^i(G^\gamma)$ set of rationalizable profiles for G^γ.

Proposition 1. $\mathcal{R}^i(G) \subseteq \cup\{R^i(G^\gamma) : \gamma \in \Delta\} \subseteq \tilde{\mathcal{R}}^i(G)$.

Proposition 2. If A^i is a non-empty convex subset of a finite dimensional space and u^i is strictly concave in a^i, then for all $i \in N$

$$\mathcal{R}^i(G) = \cup\{R^i(G^\gamma) : \gamma \in \Delta\} = \tilde{\mathcal{R}}^i(G).$$

3. Equilibria and rationalizability

Definition 2. a) a^* is an equilibrium for G if for all $i \in N$, $a^{*i} \in \mathcal{B}_{A^i}(a^{*-i})$.
b) a^* is a weak equilibrium for G if for all $i \in N$, $a^{*i} \in \tilde{\mathcal{B}}_{A^i}(a^{*-i})$.

$\mathcal{E}(G)$ ($\tilde{\mathcal{E}}(G)$) set of equilibria (weak equilibria) of G

Proposition 3. $\mathcal{E}(G) \subseteq \cup\{E(G^\gamma) : \gamma \in \Delta\} \subseteq \tilde{\mathcal{E}}(G)$.
As a consequence $\mathcal{E}(G) \subseteq \cup\{E(G^\gamma) : \gamma \in \Delta\} \subseteq \cup\{R(G^\gamma) : \gamma \in \Delta\} \subseteq \tilde{\mathcal{R}}(G)$.

Proposition 4. $\mathcal{E}^i(G) \subseteq \mathcal{R}^i(G)$, $\tilde{\mathcal{E}}^i(G) \subseteq \tilde{\mathcal{R}}^i(G)$.
And $\mathcal{E}(G) \subseteq \mathcal{R}(G)$, $\tilde{\mathcal{E}}(G) \subseteq \tilde{\mathcal{R}}(G)$. The inclusions can be strict.

4. A game with vector-valued utilities and finite strategy sets

Payoffs	b_1	b_2	b_3	b_4	b_5	b_6
a_1	3, (1,3)	3, (2,3)	0, (3,2)	0, (4,1)	1, (4,0)	2, (5,0)
a_2	0, (1,3)	0, (2,3)	3, (3,2)	3, (4,1)	1, (4,0)	2, (5,0)
a_3	2, (1,3)	1, (2,3)	1, (3,2)	2, (4,1)	0, (4,0)	0, (5,0)

Sets of rationalizable and weak rationalizable strategies for the vector-valued game G
$\mathcal{R}^1(G) = \{a_1, a_2\}$, $\mathcal{R}^2(G) = \{b_2, b_3, b_4, b_6\}$, $\tilde{\mathcal{R}}^1(G) = \{a_1, a_2\}$, $\tilde{\mathcal{R}}^2(G) = \{b_1, b_2, b_3, b_4, b_5, b_6\}$.
Sets of rationalizable strategies for the weighted maxmin games G^γ

$$R^1(G^\gamma) = \begin{cases} \{a_1\} & if \quad \alpha < \frac{1}{2} \\ \{a_2\} & if \quad \frac{1}{2} < \alpha < 1 \\ \{a_1, a_2\} & if \quad \alpha = \frac{1}{2} \, or \, \alpha = 1 \end{cases}, \quad R^2(G^\gamma) = \begin{cases} \{b_1, b_2\} & if \quad \alpha \leq \frac{1}{4} \\ \{b_2\} & if \quad \frac{1}{4} < \alpha < \frac{1}{2} \\ \{b_2, b_3\} & if \quad \alpha = \frac{1}{2} \\ \{b_3\} & if \quad \frac{1}{2} < \alpha < \frac{3}{4} \\ \{b_3, b_4\} & if \quad \alpha = \frac{3}{4} \\ \{b_4\} & if \quad \frac{3}{4} < \alpha < 1 \\ \{b_6\} & if \quad \alpha = 1 \end{cases}.$$

$b_1 \in \cup\{R^2(G^\gamma) : \gamma \in \Delta\} \setminus \mathcal{R}^2(G)$. and $b_5 \in \tilde{\mathcal{R}}^2(G) \setminus \cup\{R^2(G^\gamma) : \gamma \in \Delta\}$.

Sets of equilibria and weak equilibria of G

$\mathcal{E}(G) = \{(a_1, b_2), (a_2, b_3), (a_2, b_4), (a_1, b_6), (a_2, b_6)\}$, $\tilde{\mathcal{E}}(G) = \mathcal{E}(G) \cup \{(a_1, b_1), (a_1, b_5), (a_2, b_5)\}$

The union of the sets of equilibria of G^γ

$$\cup\{E(G^\gamma) : \gamma \in \Delta\} = \{(a_1, b_1), (a_1, b_2), (a_2, b_3), (a_2, b_4), (a_1, b_6), (a_2, b_6)\}.$$

The union of the sets of rationalizable strategies of G^γ

$$\cup\{R(G^\gamma) : \gamma \in \Delta\} = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_2), (a_2, b_3), (a_2, b_4), (a_1, b_6), (a_2, b_6)\}.$$

There is no inclusion relation between $\mathcal{R}(G)$ and $\cup\{E(G^\gamma) : \gamma \in \Delta\}$, since $(a_1, b_1) \in \cup\{E(G^\gamma) : \gamma \in \Delta\} \setminus \mathcal{R}(G)$ and $(a_1, b_4) \in \mathcal{R}(G) \setminus \cup\{E(G^\gamma) : \gamma \in \Delta\}$.
Note that (a_1, b_3) is composed by rationalizable strategies, and therefore by weak rationalizable strategies, but it is neither an equilibrium nor a weak equilibrium of G.

5. A game with vector-valued utilities and infinite strategy sets

Two individuals have access to a finite common-pool resource, with q^i the units for player i. Benefit of player i: $\quad u_i(q^1, q^2) = \max\{q^i(1 - (q^1 + q^2)), 0\}$.
Same vector-valued utility function $u : A \to \mathbb{R}_+^2$, $u := (u_1, u_2)$, where $A^i = [0, 1]$, $A = A^1 \times A^2$. Game $G = \{(A^i, u)_{i=1,2}\}$.
Equilibria and weak equilibria sets

$$\mathcal{E}(G) = \left\{(q^1, q^2) : 0 < q^1 \leq \frac{1}{2} - \frac{1}{2}q^2, 0 < q^2 \leq \frac{1}{2} - \frac{1}{2}q^1\right\} \cup \left\{(\frac{1}{2}, 0), (0, \frac{1}{2})\right\}$$

$$\tilde{\mathcal{E}}(G) = \left\{(q^1, q^2) : 0 \leq q^1 \leq \frac{1}{2} - \frac{1}{2}q^2, 0 \leq q^2 \leq \frac{1}{2} - \frac{1}{2}q^1\right\}.$$

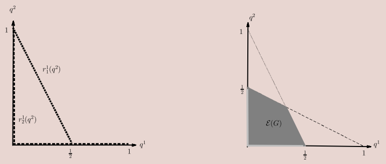

Figure 1: *Best responses of player 1 and equilibria of G.*

For weighted maxmin games, $\gamma^1 = (\alpha^1, 1 - \alpha^1)$ and $\gamma^2 = (\alpha^2, 1 - \alpha^2)$, with $\alpha^1, \alpha^2 \in [0, 1]$.
Thus, if $q^1 + q^2 \leq 1$ the payoff of player i in G^γ is

$$w_\gamma^i(q^1, q^2) = \min\left\{q^i \frac{1 - (q^1 + q^2)}{\alpha^i}, q^i \frac{1 - (q^1 + q^2)}{1 - \alpha^i}\right\}.$$

Otherwise, the payoff is null.

Maxmin best response function.
For $\alpha^i = 0$, $r_\gamma^i(q^j) = r_2^i(q^j)$
For $\alpha^i = 1$, $r_\gamma^i(q^j) = r_1^i(q^j)$.
For $\alpha^i \in (0, 1)$, when $q^j = 0$, $r_\gamma^i(q^j) = [0, 1]$.
In other case,

$$r_\gamma^i(q^j) = \begin{cases} \frac{1}{2} - \frac{1}{2}q^j & if \quad q^j > \frac{1-\alpha^i}{1+\alpha^i} \\ \frac{\alpha^i}{1-\alpha^i}q^j & if \quad 0 < q^j \leq \frac{1-\alpha^i}{1+\alpha^i} \\ [0, 1] & if \quad q^j = 0 \end{cases}.$$

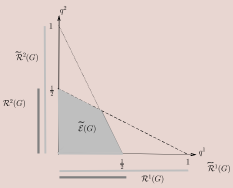

Figure 2: *Rationalizable strategies and weak equilibria of G.*

$$[0, \frac{1}{2}] = \mathcal{R}^i(G) \subset \cup\{R^i(G^\gamma) : \gamma \in \Delta\} = \tilde{\mathcal{R}}^i(G) = [0, 1], i = 1, 2.$$

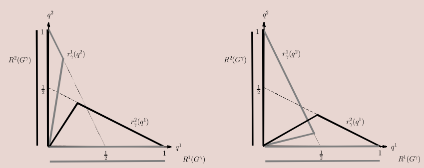

Figure 3: *Maxmin best responses and rationalizable strategies of G^γ.*

Any rationalizable strategy for a player is part of at least an equilibrium for G, but there are weak rationalizable strategies which are not part of a weak equilibrium.

6. References

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Acknowledgement: Grant PID 2021-124981NB-I00 funded by MICIU/AEI/ 10.13039/501100011033 and by "ERDF/EU", and Grant RED2022-134540-T funded by MICIU/AEI/10.13039/501100011033.

I IBERIAN CONFERENCE ON MCDM/A. May 8 to 9, 2025. Coimbra, Portugal