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A cooperative game theory approach to cost sharing in capacity synthesis problems. A bi-criteria model.

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1. ABSTRACT

This paper considers networks in which nodes have communication needs between them, that is, any pair of agents, located at the nodes of the network, requests a connection with a certain capacity. The cost of building, maintaining or using an edge with a given capacity is the same across any pair of agents. It is known that feasibility is reached by any Maximal-Capacity Spanning Tree (MCST). The capacity that any pair of agents requires is considered fulfilled if and only if, in the resulting tree, there exists a path between them such that every edge provides at least this required capacity. Hence, a *benefit capacity synthesis cooperative game* is defined, in which the worth of each coalition S is the sum of the capacities of the edges in a MCST on S. On the other hand, the agents have to pay for the link of a MCST. Therefore a *cost capacity synthesis cooperative game* is defined, in which the worth of each coalition S is the cost of the links which ensures the capacities required for the agents in S with respect to all the agents in the network. These two games have been already studied in previous works. In this paper we extend the idea of stability to a bi-dimensional framework and analyze stable allocations with respect to both games . We also relate them with some wellknown solution concepts in the literature.

2. Capacity synthesis problem

In the *capacity synthesis* problem, a set of agents share a network for exchanging information, for transporting commodities along roads or shipping channels, for establishing commercial channels with limited capacity of human or material resources, etc. For capacity synthesis problems, a required capacity between any pair of agents (bandwidth, road-width, depth of the channel, etc.) is needed. A feasible graph for the capacity synthesis problem is one in which any pair of agents is connected by a path whose edges all hold the capacity of at least the required capacity between them. Among these feasible graphs, we are interested on those with minimum cost. It is well known that a feasible network with minimum total cost is obtained by using a *maximal spanning tree* with respect to the capacities.

Example

Consider $N = \{1,2,3,4\}$, *and* $t = \begin{pmatrix} 0 & 5 & 5 & 3 \\ 5 & 0 & 1 & 5 \\ 5 & 1 & 0 & 6 \\ 3 & 5 & 6 & 0 \end{pmatrix}$

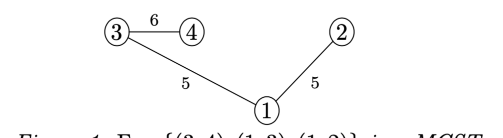


Figure 1: $\Gamma = \{(3,4),(1,3),(1,2)\}$ *is a MCST.*

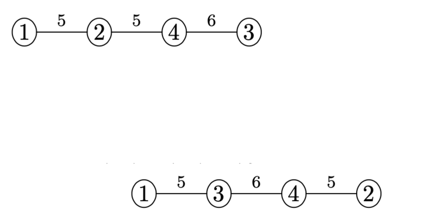


Figure 2: Other possible MCSTs

3. How to share $V(N,t)$

A solution, f, for the cost sharing problem, (N,t), assigns to each problem, (N,t), a subset of $I^*(N,t)$, $f(N,t)$. For instance, the celebrated Bird solution for the minimum cost spanning tree problem can be adapted to this framework to provide a single-valued solution concept as follows: Select a maximal spanning tree and an arbitrary agent to be the non-paying "source", then charge to every other agent the cost of its adjacent upstream edge.The Bird rule, denoted by B is the uniform average of the (Γ, i)-Bird solutions over all maximal spanning trees and all agents.

Example (continued)

	Agent 1	*Agent 2*	*Agent 3*	*Agent 4*
$\Gamma = \{(3,4),(1,3),(1,2)\}$				
$B^{(\Gamma,1)}(N,t)$	0	5	5	6
$B^{(\Gamma,2)}(N,t)$	5	0	5	6
$B^{(\Gamma,3)}(N,t)$	5	5	0	6
$B^{(\Gamma,4)}(N,t)$	5	5	6	0

	Agent 1	*Agent 2*	*Agent 3*	*Agent 4*
$\Gamma' = \{(1,2),(2,4),(3,4)\}$				
$B^{(\Gamma',1)}(N,t)$	0	5	6	5
$B^{(\Gamma',2)}(N,t)$	5	0	6	5
$B^{(\Gamma',3)}(N,t)$	5	5	0	6
$B^{(\Gamma',4)}(N,t)$	5	5	6	0

	Agent 1	*Agent 2*	*Agent 3*	*Agent 4*
$\Gamma'' = \{(1,3),(3,4),(2,4)\}$				
$B^{(\Gamma'',1)}(N,t)$	0	5	5	6
$B^{(\Gamma'',2)}(N,t)$	5	0	6	5
$B^{(\Gamma'',3)}(N,t)$	5	5	0	6
$B^{(\Gamma'',4)}(N,t)$	5	5	6	0

	Agent 1	*Agent 2*	*Agent 3*	*Agent 4*
$B(N,t)$	15/4	15/4	17/4	17/4

Figure 3: The Bird rule.

Cooperative game theory provides tools to deal with cost sharing problem associated to the capacity synthesis problem. We consider two different cooperative games, a *cost game* denoted by (N,c), and another different game, that we call *benefit game*, denoted by (N,v).

In the cost game, $c(S) = V(N,t^S)$ for every $S \subseteq N$, where $t^S_{ij} = 0$ if $(i,j) \in (N \setminus S)(2)$, and $t^S_{ij} = t_{ij}$ otherwise.

On the other hand, in the benefit game, $v(S) = V(S,t)$ for every $S \subseteq N$, where (S,t) is the projection of (N,t) on S.

Notice that $c(N) = v(N) = V(N,t)$, and $v(S) \leq c(S)$ for all $S \subseteq N$.

The requirements $x(S) \leq c(S)$ for all $S \subseteq N$ are seen as the usual core stability property. We denote the core of the game (N,c) by $C(N,c)$. On the other hand, $v(S) \leq x(S)$ for all $S \subseteq N$, apart from being the stability conditions in the benefit game, can be also seen as a normative requirement of fairness if we think in terms of costs. Analogously $C(N,v)$ denote the set of stable allocations for (N,v).

Proposición 3.1 *Allocations provided by the Bird rule belong to* $C(N,c) \cap C(N,v)$.

Example (continued)

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$c(S)$	13	11	12	14	15	16	16	16	16	16	16	16	16	16	16
$B(S)$	15/4	15/4	17/4	17/4	15/2	8	8	8	8	17/2	47/4	47/4	49/4	49/4	16
$v(S)$	0	0	0	0	5	5	3	1	5	6	10	10	11	11	16

Define, for each $S \subset N$, $e^v(x,S) = v(S) - x(S)$, $e^c(x,S) = x(S) - c(S)$ *and the vector-valued excess function* $e(x,S) = (e^v(x,S), e^c(x,S))^t$.

Definición 3.1 *The set of preference core allocations for the capacity synthesis problem* (N,t) *is* $PC(N,t) = \{x \in \mathbb{R}^n : e(x,S) \leqq \mathbf{0}, \forall S \subseteq N\}$.[1]

Example (continued)

$PC(N,c)$ is the convex hull of $P_1 = (5,0,5,6)^t$, $P_2 = (5,0,6,5)^t$, $P_3 = (0,5,5,6)^t$, $P_4 = (0,5,6,5)^t$, $P_5 = (5,5,6,0)^t$ and $P_6 = (5,5,0,6)^t$, and $B(N,t) = 1/4(P_1 + P_4 + P_5 + P_6)$.

Definición 3.2 *The set of preference p-core allocations for the capacity synthesis problem* (N,t) *is* $PpC(N,t) = \{x \in \mathbb{R}^n : e(x,S) \leqq p, \forall S \subseteq N\}$.

For each $x \in I^*(N,t)$, denote $\bar{p}^v(x) = max_{S \subset N} e^v(x,S)$, $\bar{p}^c(x) = max_{S \subset N} e^c(x,S)$ and $\bar{p}(x) = (\bar{p}^v(x), \bar{p}^c(x))^t$.

Definición 3.3 *The generalized least core for the capacity synthesis problem* (N,t) *is*

$$GLC(N,t) = \{x \in \mathbb{R}^n : \nexists y \in I^*(N,t) \text{ such that } \bar{p}(y) \leqq \bar{p}(x),\ \bar{p}(y) \neq \bar{p}(x)\}.$$

Notice that, if $x \in GLC(N,t)$, then $(x, \bar{p}(x))$ is a non-dominated solution of the following bi-criteria linear programming problem:

$$\begin{array}{ll} min & (p^v, p^c) \\ s.t.: & v(S) - x(S) \leq p^v \ \ \forall S \subset N \\ & x(S) - c(S) \leq p^c \ \ \forall S \subset N \\ & x(N) = V(N,t) \\ & x \geqq \mathbf{0}, \end{array} \tag{3.1}$$

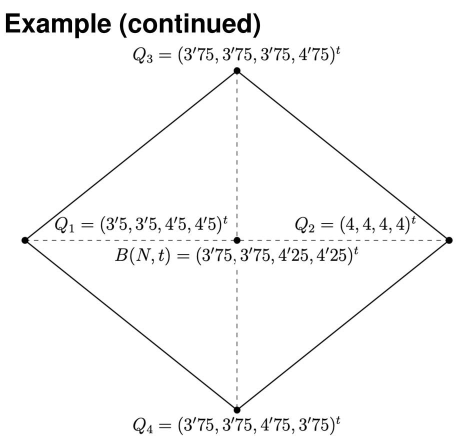
and conversely, if (x^*, p^*) is a non-dominated solution of the above multi-criteria linear programming problem, then $x^* \in GLC(N,t)$ and $\bar{p}(x^*) = p^*$.

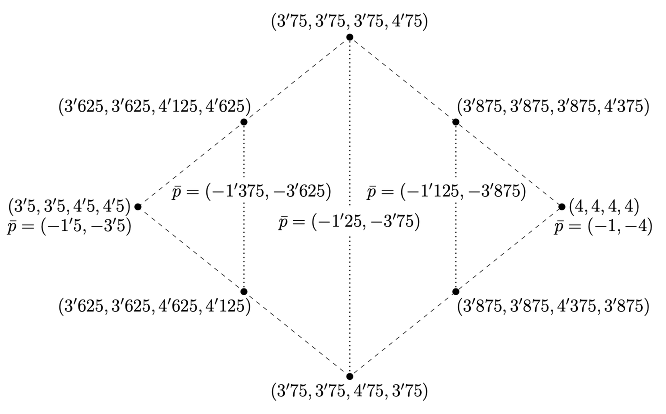
For each $x \in I^*(N,t)$, consider the $2 \times (2^n - 2)$-matrix, $E(x)$, whose two rows are $(e^v(x,S))_{S \subset N}$ and $(e^c(x,S))_{S \subset N}$, arranged in order of decreasing magnitude. Denote by $E^k(x)$, $k = 1,2,\ldots 2^n - 2$, the column vectors of $E(x)$. We say that $y \succ x$, if $E(y) \leq_{lex} E(x)$, that is, $E^k(y) \leq E^k(x)$ for the first column, k, in which $E(y)$ and $E(x)$ are different.[2]

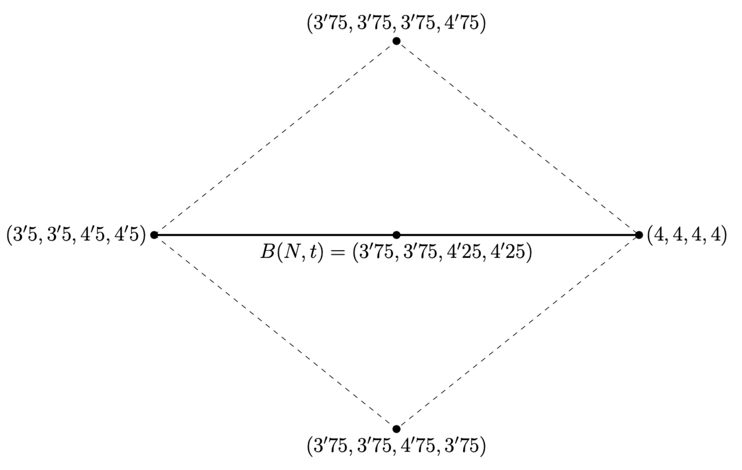
Definición 3.4 *The generalized nucleolus for the capacity synthesis problem* (N,t) *is:* $GN(N,t) = \{x \in \mathbb{R}^n : \nexists y \in I^*(N,t)$ *such that* $y \succ x\}$.

The generalized nucleolus can be obtained by solving multi-criteria linear programming programs in a recursive procedure.[3]

Example (continued)







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[1] $\mathbf{0} = (0,0)^t \in \mathbb{R}^2$ and $e(x,S) \leqq \mathbf{0}$ means $e^v(x,S) \leq 0$ and $e^c(x,S) \leq 0$

[2] $E^k(y) \leq E^k(x)$ means $E^k(y) \leqq E^k(x)$ and $E^k(y) \neq E^k(x)$.

[3] See Hinojosa M.A., Mármol A.M. and Thomas L.C. (2005), "Core, least core and nucleolus for multiple scenario cooperative games", European Journal of Operational Research, 164, 225-238