Interactive Multiobjective Optimization: Trading Off and Desirable Properties

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Outline

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Multiobjective Optimization (MOP)

Types of MOP Methods (Hwang & Masud, 1979; Miettinen, 1999)

- DM specifies preferential information before the solution process $\Rightarrow$ A priori methods.
- DM specifies preferential information after the solution process $\Rightarrow$ A posteriori methods.
- DM provides preferential information iteratively during the solution process $\Rightarrow$ Interactive methods.

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Interactive Methods

Potential Advantages

- DM can iteratively specify and modify preferences. No pre-fixed preference structure is assumed.
- The amount of information and solutions to be considered at a time is low. Decrease cognitive load.
- During an interactive solution process, the DM can learn about (Belton et al., 2008):
  - ✓ The interdependencies among the objectives;
  - ✓ What kind of solutions are attainable;
  - ✓ One's own preferences.
- Many interactive methods have been developed, using different types of preferential information.

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- Most methods deal with Pareto optimal solutions.
- DM must allow sacrifice in some objective(s) to gain improvement in others.
- Gardiner & Vanderpooten (1997): the median number of iterations reported in the interactive solution processes has been between three and eight.
- Possible reasons:
  - ✓ Anchoring: the starting point of an interactive method matters (Buchanan & Corner, 1997).
  - ✓ Trade-off conflict is a major source of decisional stress (Janis & Mann, 1977).
  - ✓ Choice sets that are high in trade-off conflict lead to less accurate decision making (Aloysius et al., 2006).
  - ✓ People do not react symmetrically to gains and losses (Korhonen & Wallenius, 1996).
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The NAUTILUS family of Interactive Methods

So, why not starting at a bad (inferior) solution

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So, why not starting at a bad (inferior) solution
and improving every objective function at each iteration?

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First part of this talk

Features of Interactive Methods

Common Structure

1. Generate one (or several) initial (efficient) solution.
2. Present the current solution(s) to the DM
3. Is the DM satisfied with the solution?
   - ✓ "Yes": end.
   - ✓ "No": go to step 4
4. Ask the DM for new preference information
5. Generate new (efficient) solution(s)
6. Go to step 2

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Features of Interactive Methods (cont.)

Differences - Type of preferential information

1. Comparison of solutions.
2. Local trade-offs.
3. Aspiration levels.
4. Classification.
5. Weights.
6. Bounds.
7. Directions of improvement.

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Two Phases

- Learning phase:
  - ✓ exploring,
  - ✓ finding an area of interest.
- Decision phase:
  - ✓ fine-tuning,
  - ✓ finding final solution.

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Assessing Performance of Interactive Methods

OK, we've got a method, but... is it any good?

- How to decide whether an interactive method is "good" or not?
- Which features are desirable for a "good" interactive method?
- What types of assessments have been done in the literature?
- Can different desirable features be advised for each of the two phases (learning - decision)?
- Setting and carrying out tests.
- Can a combination of methods of different types work better?

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Choosing an interactive method: why and when

Second part of this talk

Basic Concepts and Notations

The Multiobjective Optimization Problem

$$(MOP) \quad \begin{cases} \min & \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_k(\mathbf{x}))^T \\ s.t. & \mathbf{x} \in S. \end{cases}$$

- $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ is a decision vector;
- $S \subset \mathbb{R}^n$ is the feasible set;
- $\mathbf{f}(S) \subset \mathbb{R}^k$ is the feasible objective set;
- $E \subset S$ is the set of Pareto optimal decision vectors;
- $\mathbf{f}(E) \subset \mathbf{f}(S)$ is the Pareto optimal set in the objective space;
- $\mathbf{z}^* = (z_1^*, z_2^*, \dots, z_k^*)^T$, $z_i^* = \min_{\mathbf{x} \in E} f_i(\mathbf{x})$, is the ideal objective vector;
- $\mathbf{z}^{**} = (z_1^* - \varepsilon, z_2^* - \varepsilon, \dots, z_k^* - \varepsilon)^T$, is an utopian objective vector;
- $\mathbf{z}^{nad} = (z_1^{nad}, z_2^{nad}, \dots, z_k^{nad})^T$, $z_i^{nad} = \max_{\mathbf{x} \in E} f_i(\mathbf{x})$, is the nadir objective vector.

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Basic Concepts and Notations

The Multiobjective Optimization Problem

$$(MOP) \quad \begin{cases} \min & \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_k(\mathbf{x}))^T \\ s.t. & \mathbf{x} \in S. \end{cases}$$

- $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ is a decision vector;
- $S \subset \mathbb{R}^n$ is the feasible set;
- $\mathbf{f}(S) \subset \mathbb{R}^k$ is the feasible objective set;
- $E \subset S$ is the set of Pareto optimal decision vectors;
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Reference Points and Achievement Functions

- $\mathbf{q} = (q_1, x_2, \ldots, q_k) \in \mathbb{R}^k$ is a reference point, formed by aspiration (desired) levels for the objective functions;
- $\mathbf{q}$ is said to be an achievable reference point if $\mathbf{q} \in \mathbf{f}(S) + \mathbb{R}^k_+$, that is, either $\mathbf{q}$ is a feasible objective vector, or it is dominated by some Pareto optimal vector(s).
- Given $\mathbf{q}$, a Pareto optimal solution is obtained by solving the following problem:

$$(RP) \quad \begin{cases} \min & s_i(\mathbf{x}, \mathbf{q}) = \max\limits_{i=1,\ldots,k} \{\mu_i(f_i(\mathbf{x}) - q_i)\} + \rho \sum\limits_{i=1}^{k} \dfrac{f_i(\mathbf{x}) - q_i}{z_i^{nad} - z_1^{**}} \\ s.t. & \mathbf{x} \in S. \end{cases}$$

✓ $s_i(\mathbf{x}, \mathbf{q})$ is called an achievement scalarizing function;
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The NAUTILUS Family

Main features

- Start from an inferior solution (nadir or stated by the DM).
- Simultaneously improve all the objective functions at each iteration.
- Reach the Pareto optimal set at the last iteration.
- No trade-offs.
- At each iteration:
  - ✓ What is still achievable?
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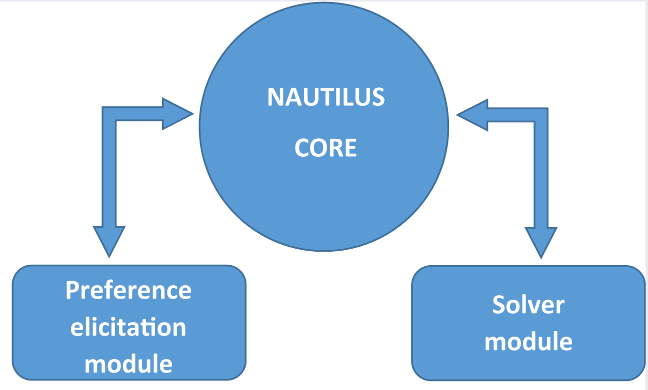
The NAUTILUS Framework

Miettinen and Ruiz (2016)



The NAUTILUS core

A Basic NAUTILUS Iteration

- Before starting:
  - ✓ We choose the initial (inferior) point $\mathbf{z}^{nad}$.
  - ✓ The DM specifies the desired number of iterations, *itn*.
- At each iteration h:
  - ✓ $\mathbf{z}^h$ is the current iteration point in the objective space ($\mathbf{z}^0 = \mathbf{z}^{nad}$).
  - ✓ it^h is the number of remaining iterations, ($it^0 = itn$).
- Iteration h:
  - ✓ Given $\mathbf{z}^{h-1}$, find a Pareto optimal solution $\mathbf{x}^h$, such that $\mathbf{f}^h = \mathbf{f}(\mathbf{x}^h)$ dominates $\mathbf{z}^{h-1}$.
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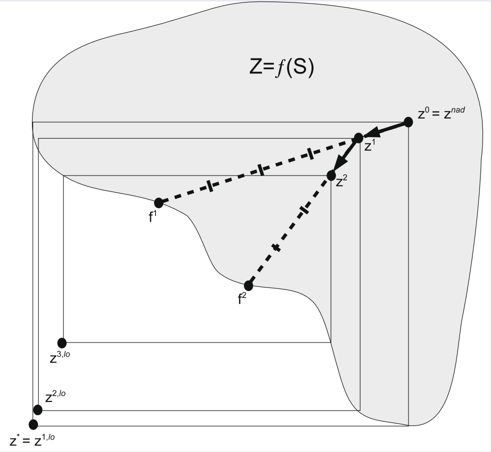

A few more things

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- Further options: DM can decide to:
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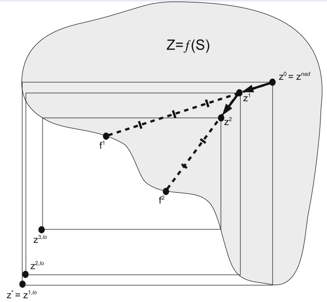

Calculating $z_i^{h,lo}$ and $z_i^{h,up}$

- Solve problems:

$$(P_r^h) \quad \begin{cases} \min & f_r(\mathbf{x}) \\ s.t. & f_i(\mathbf{x}) \leq z_i^{h-1}, \\ & (i = 1, \ldots, k,\ i \neq r) \\ & \mathbf{x} \in S. \end{cases}$$

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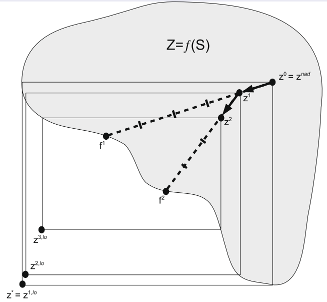

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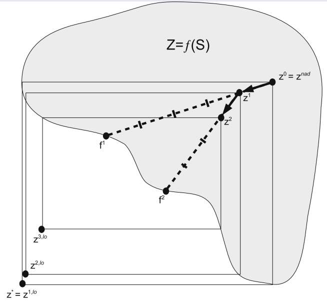

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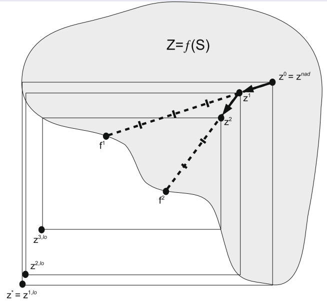

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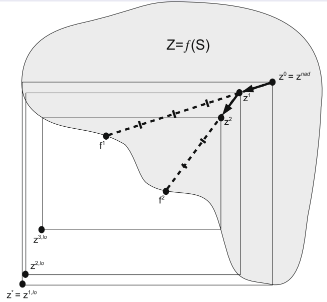

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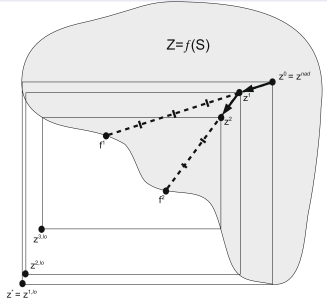

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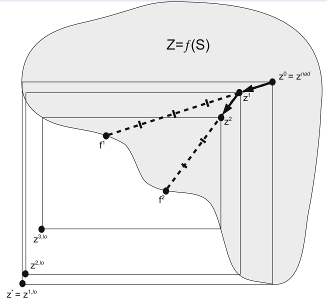

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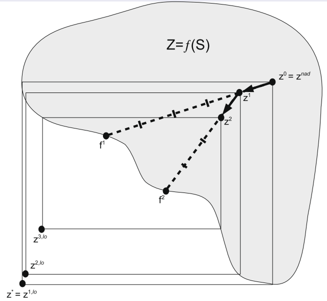

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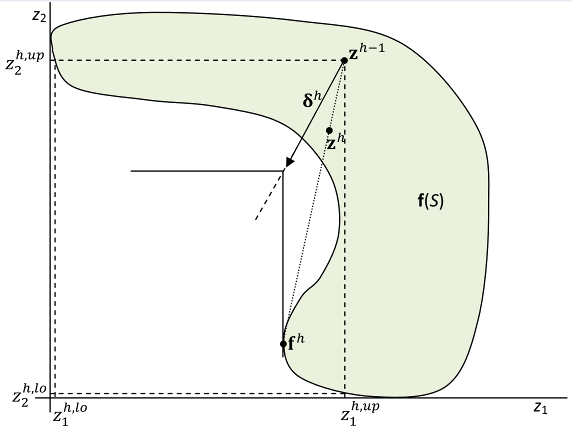
Preference Elicitation Module

Two options

- Direction of simultaneous improvement.
- Choice of one among several alternatives.

Preference Elicitation Module

Simultaneous Improvement



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Eliciting δ^h.

- Direct specification.
- Importance ranks J_r ($r = 1, \ldots, s$):

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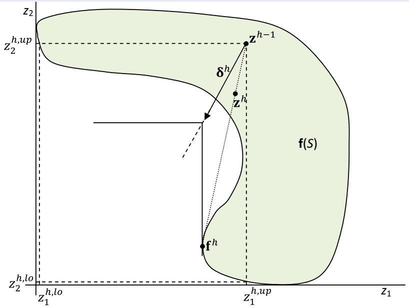

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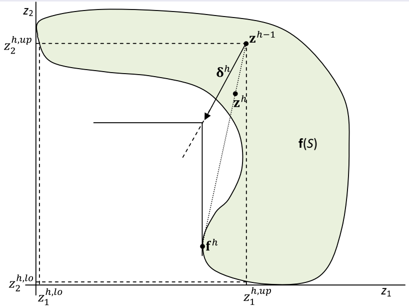

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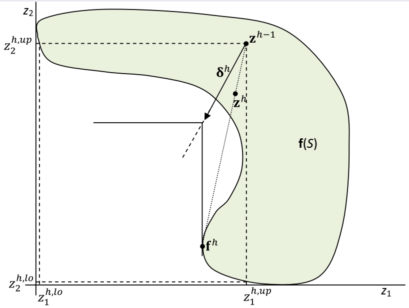

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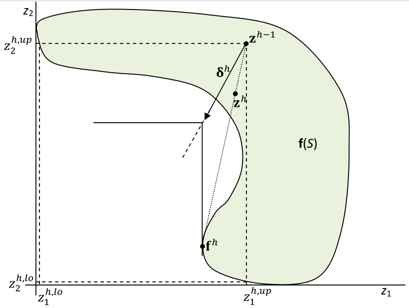

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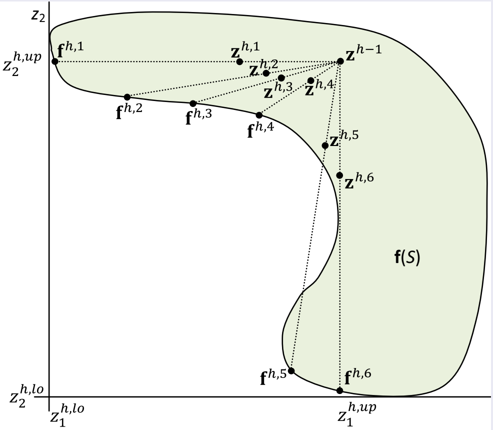

Solver Module

Again, two options

- Optimization option. Problems (RP) and (P_r^h) are solved at each iteration using an appropriate single objective optimization method.
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The NAUTILUS family

Existing NAUTILUS Variants

		Preference elicitation	
		Direction of improvement	Choose one solution
Solver	Optimization option	NAUTILUS (1) Miettinen et al. (2010) NAUTILUS 2 Miettinen et al. (2015)	
	A posteriori option		E-NAUTILUS Ruiz et al. (2015)

NAUTILUS Navigator

Ruiz et al. (2019)

NAUTILUS Navigator

Choosing an Interactive Method

How to Assess an Interactive Mehtod

- What has been done in interactive methods assessment?
- Who has assessed the methods (human DM, artificial DM, utility functions)?
- What aspects have been considered?
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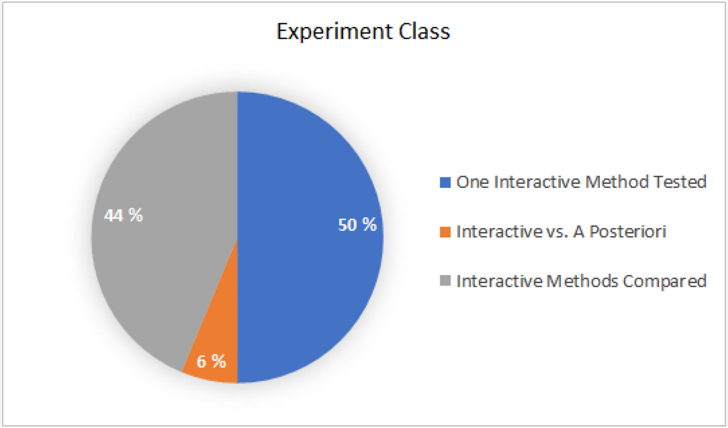
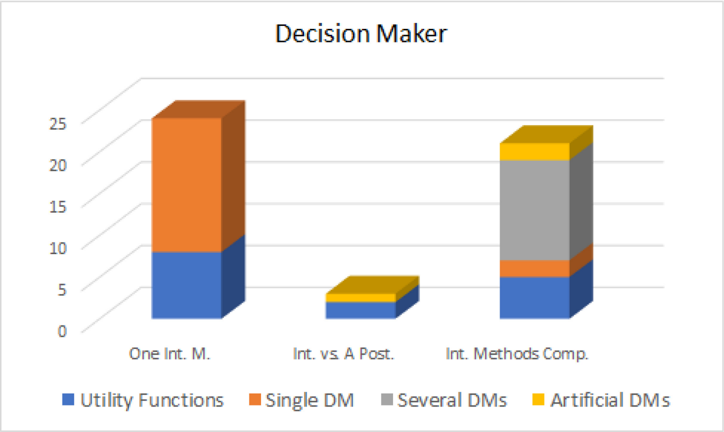
Experiment Class



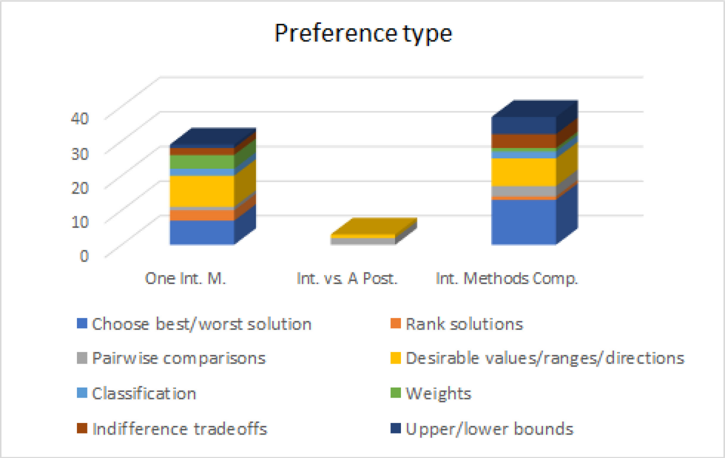
Types of Experiments

Decision Maker



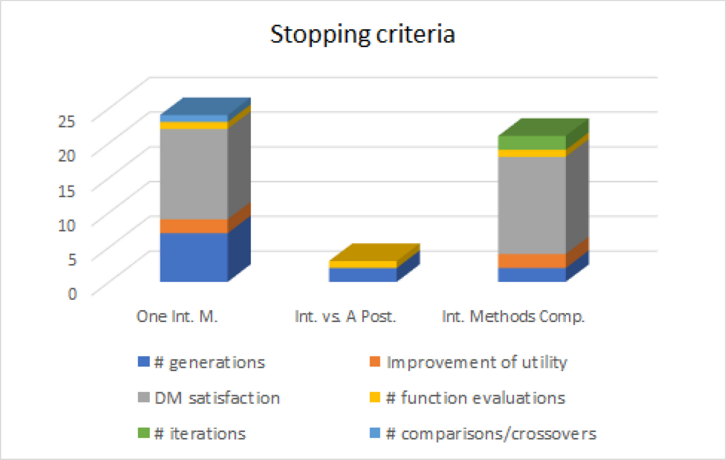
Types of Experiments

Preference Type



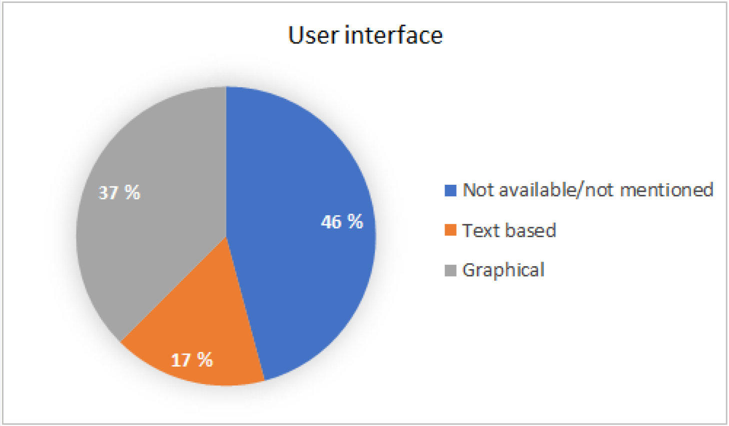
Types of Experiments

Stopping Criteria



Types of Experiments

User Interface



Desirable properties

General Properties

- **GP_1** The method captures the preferences of the DM.
- **GP_2** The method sets as low cognitive burden on the DM as possible.
- **GP_3** A user interface supports the DM in problem solving.
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- **LP_1** The method helps the DM avoid anchoring.
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- **DP$_2$** The method reaches the DM's most preferred solution.
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What has been done - Yet to be done

To summarize...

Properties	# of experiments	Human DMs	UFs	Artificial DMs
GP_1 (Capturing preferences)	2	✓	✓	✓
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GP_3 (User interface)	-	✓		
GP_4 (Being in control)	-	✓		
GP_5 (Early termination)	-	✓	?	?
LP_1 (Anchoring)	1	✓	✓	✓
LP_2 (Exploring PO)	-	✓	✓	✓
LP_3 (Changing area)	-	✓	?	✓
LP_4 (Learning)	2	✓		✓
LP_5 (Uncertain preference)	-	✓	✓	✓
LP_6 (Region of interest)	-	✓	✓	✓
DP_1 (Convinced)	6	✓		
DP_2 (MPS)	18	✓	✓	✓
DP_3 (Iterations / waiting time)	8 / 10	✓	✓	✓
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Design of empirical experiments (Afsar et al., 2023)

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Afsar et al. (2024)

- Methods compared:
  - ✓ E-NAUTILUS. Trade-off free.
  - ✓ Reference Point Method (RPM, Wierzbicki, 1980)
  - ✓ NIMBUS (Miettinen and Mäkelä, 2006). Classification.
- Participants
  - ✓ 164 students Faculty of Economy and Business Administration, UMA.
  - ✓ Divided into 3 homogeneous groups. Each group tested one method.
- Problem related to the sustainability of the country.
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  - ✓ NIMBUS (Miettinen and Mäkelä, 2006). Classification.
- Participants
  - ✓ 164 students Faculty of Economy and Business Administration, UMA.
  - ✓ Divided into 3 homogeneous groups. Each group tested one method.
- Problem related to the sustainability of the country.
- Use of a Graphical User Interface.

The First Experiment

Afsar et al. (2024)

- Methods compared:
  - ✓ E-NAUTILUS. Trade-off free.
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Questionnaire Design

Research questions

- **RQ1**: Cognitive load: How extensive is the cognitive load of the whole solution process?
  - ✓ 1) “The method sets as low cognitive burden on the DM as possible.”
  - ✓ 2) “The method allows the DM to fine-tune solutions in a reasonable number of iterations and/or reasonable waiting time.”
- **RQ2**: Capturing preferences and responsiveness:
- **RQ3**: Satisfaction and confidence:
- 29 questions.

Questionnaire Design

Research questions

- **RQ1**: Cognitive load:
- **RQ2**: Capturing preferences and responsiveness: How well does the method capture and respond to the DM's preferences?
  - ✓ 1) "The method captures the preferences of the DM."
  - ✓ 2) "The DM feels being in control while interacting with the method."
  - ✓ 3) "The method easily changes the area explored as a response to a change in the preference information given by the DM."
- **RQ3**: Satisfaction and confidence:
- 29 questions.

Questionnaire Design

Research questions

- **RQ1**: Cognitive load:
- **RQ2**: Capturing preferences and responsiveness:
- **RQ3**: Satisfaction and confidence: Is the DM satisfied with the overall solution process and confident with the final solution?
  - ✓ 1) "The method allows the DM to learn about the conflict degree and tradeoffs among the objectives in each part of the Pareto optimal set explored."
  - ✓ 2) "The method does not miss any Pareto optimal solution that is more preferred (with a given tolerance) for the DM than the one chosen."
  - ✓ 3) "The method allows the DM to be fully convinced that (s)he has reached the best possible solution at the end of the solution process."
- 29 questions.

Questionnaire Design

Research questions

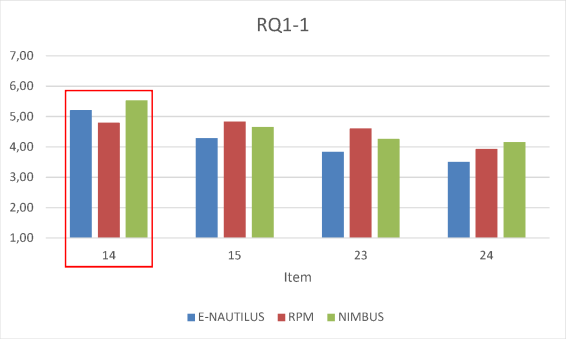
- **RQ1**: Cognitive load:
- **RQ2**: Capturing preferences and responsiveness:
- **RQ3**: Satisfaction and confidence:
- 29 questions.
  - ✓ 3 Open-ended
  - ✓ 22 Likert scale
  - ✓ 4 Likert scale + Open-ended (“Why?”)

Most Significant Results

RQ1-1: Cognitive Burden

- Satisfaction with own's performance.
- Frustration.
- Mental activity.
- Hard Work.

Results (1-7 Likert scale)



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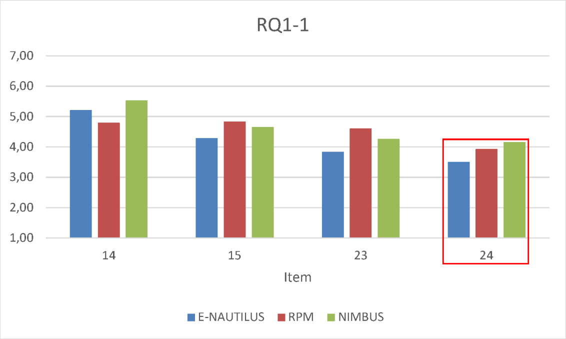

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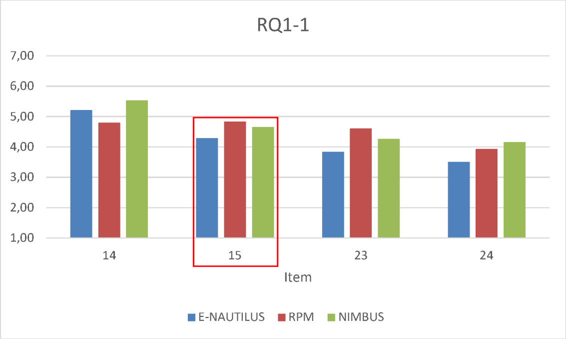

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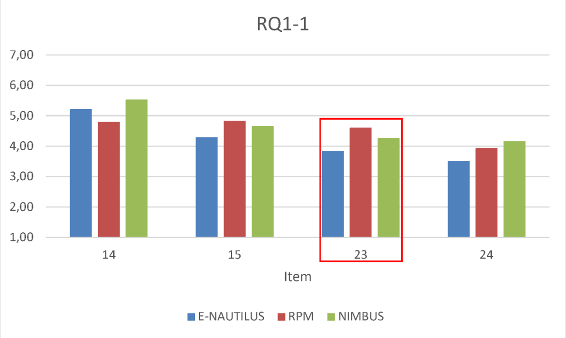

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Due to Trading-off?

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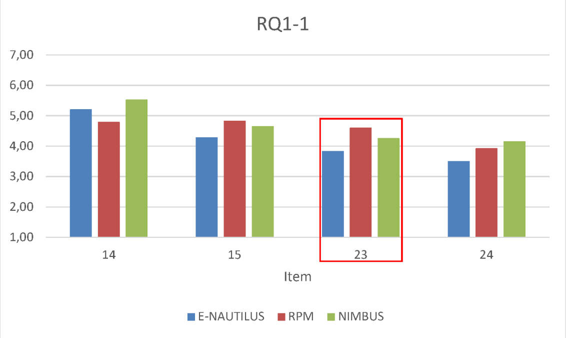
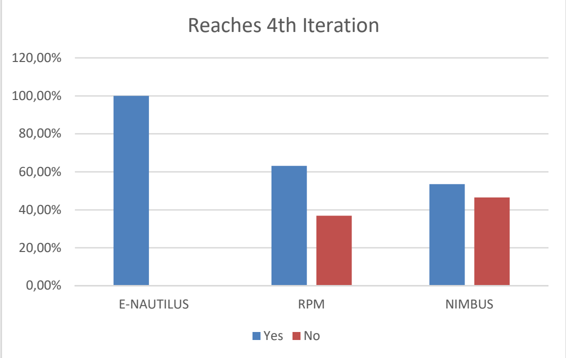


Most Significant Results

RQ2-3: Iterations/Time

- Real number of iterations.
- Real time
- Perceived iterations.
- Tiredness.

Results



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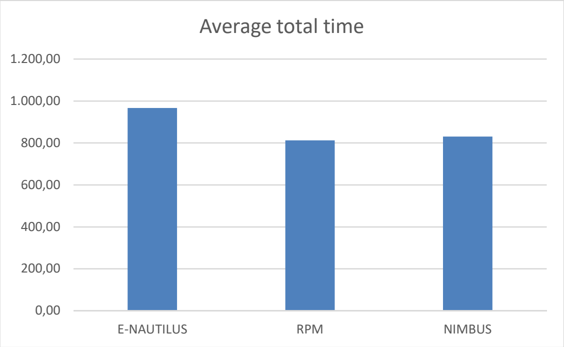

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RQ2-3: Iterations/Time

- Real number of iterations.
- Real time
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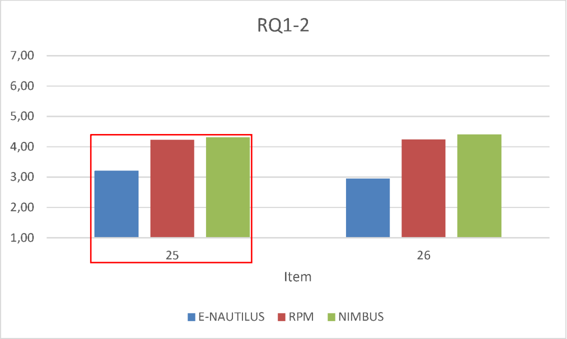
- Tiredness.

Results (1-7 Likert scale)



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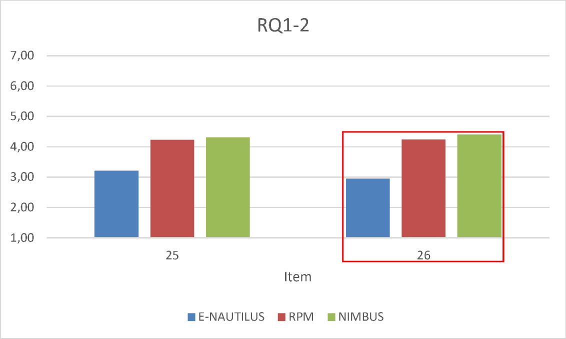
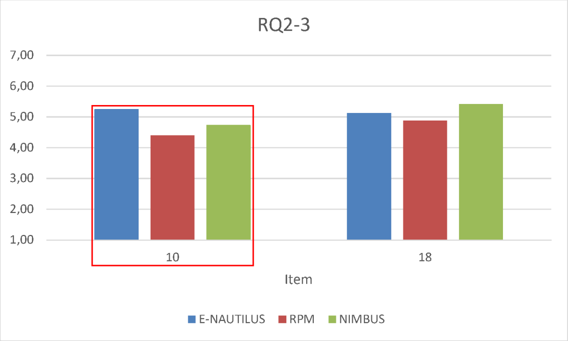


Most Significant Results

RQ1-2: Changes area explored

- Easier to explore solutions with different trade-offs.
- Reacted to preference information.

Results (1-7 Likert scale)



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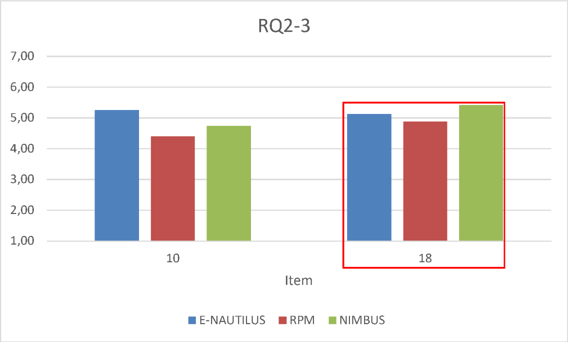

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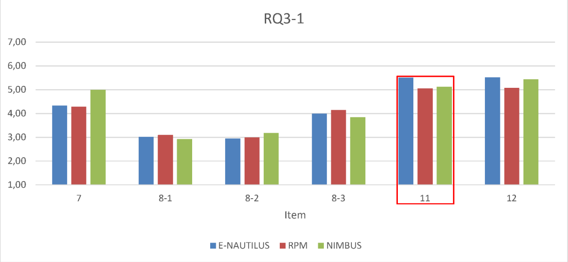
RQ3-1: Learning about trade-offs

- Global problem.
- Region of interest.
- Thinks found the best possible solution.

Results (1-7 Likert scale)



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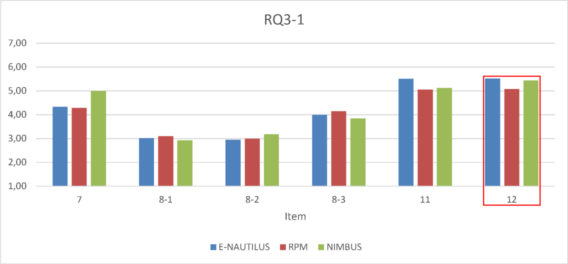

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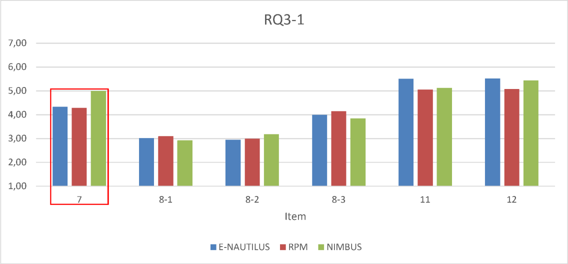
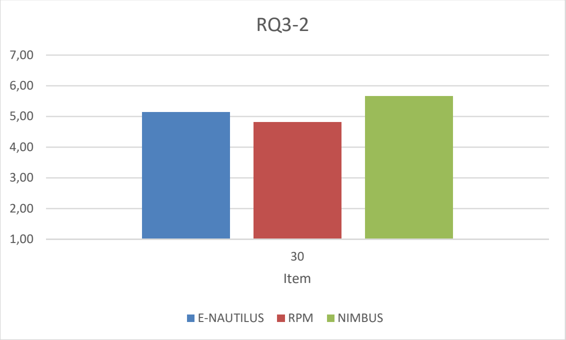


Most Significant Results

RQ3: Satisfaction and Confidence

- Satisfied with solution chosen.

Results (1-7 Likert scale)



The Second Experiment

Switching Methods

- Why this experiment?
  - ✓ Learning Phase - Decision Phase.
  - ✓ Prior experiments suggests that different methods may perform better in each phase.
- Methods used:
  - ✓ Learning Phase: NAUTILUS Navigator.
  - ✓ Decision Phase: Synchronous NIMBUS.
- Experiment Settings:
  - ✓ 48 Students.
  - ✓ Same problem as Experiment 1.
  - ✓ They all switched methods.
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Results in Short

Some Interesting Findings

- Even though some participants were not fully satisfied with their final solutions, they were convinced that no better solution could have been found either.
- The participants did learn about the problem during the process. Some participants expressed that, although they did not find a fully satisfactory solution at the end of the process, they realized that their initial expectations were too optimistic.
- Most participants generally found the process easy and not frustrating and participants' tiredness did not significantly increase.
- Participants found it easy to provide preferences using two different ways.
- Switching from one interactive method to another increased the feeling of control of the interactive solution process.
- Responses support our initial idea that the trade-off-free method selected for phase 1 was better for exploring solutions and learning about the problem, and the one in phase 2 was more suited for fine-tuning and directing the search to a satisfactory final solution.

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Conclusions

Five ideas

1. Trade-off free methods favor exploring and thus, learning.
2. It is desirable to have variants of a method using different types of preference information.
3. May NAUTILUS type methods be useful for interactive group decision making processes?
4. It is desirable to develop ways to evaluate interactive methods.
5. Different method may work better in different phases of an interactive process (learning - decision).
6. Switching methods allow the DM to be more confident on the final solution.

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Our Publications

- **Afsar, B., Miettinen, K., Ruiz, F.** (2021) *Assessing the Performance of Interactive Multiobjective Optimization Methods: A Survey*. ACM Computing Surveys, 54(4): 85.

- **Afsar, B., Silvennoinen, J., Misitano, G., Ruiz, F., Ruiz, A.B., Miettinen, K.** (2023) *Designing Empirical Experiments to Compare Interactive Multiobjective Optimization Methods*. Journal of the Operational Research Society, 74: 2327–2338.

- **Afsar, B., Silvennoinen, J., Ruiz, F., Ruiz, A.B., Misitano, G., Miettinen, K.** (2024) *An experimental design for comparing interactive methods based on their desirable properties*. Annals of Operations Research, 338: 835–856.

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- **Ruiz, A.B., Sindhya, K., Miettinen, K., Ruiz, F., Luque, M.** (2015) *E-NAUTILUS: A decision support system for complex multiobjective optimization problems based on the NAUTILUS method*. European Journal of Operational Research, 246: 218–231.

- **Ruiz, A.B., Ruiz, F., Miettinen, K., Delgado, L., Ojalehto, V.** (2019) *NAUTILUS Navigator: free search interactive multiobjective optimization without trading-off*. Journal of Global Optimization, 74: 213–231.

Obrigado!