Optimal comonotonic actuarial risk redistribution

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Motivation

- Highly correlated risks have become more and more usual in the insurance industry. Accordingly, the usual diversification effect does not apply any more.
- For instance, the global warming has led to an increase in indemnity payments in lines such as farming, household insurance, cars, etc. In fact, the frequency of droughts, floods, hurricanes and other catastrophic weather events has significantly increased.
- Other important example is the cyber-insurance, where massive cyber-attacks, or attacks with massive effects, destroy the diversification effectiveness (recall the great blackout, two weeks ago).
- Some solutions are related to the financial market,

https://www.artemis.bm/news/beazleys-cyber-cat-bond-cairney-listed-on-the-bsx/
https://www.cmegroup.com/markets/weather.html#tab_72Y0xwi=futures

- Parametric insurance is the other relevant novelty,

https://descartesunderwriting.com/
https://actuariesclimateindex.org/home/
https://docta.ucm.es/entities/publication/380c9656-b097-4cd0-b88d-24101d482fc0

- The academic actuarial approach mainly focuses on risk-sharing.
- We will propose the risk redistribution as an alternative way.
- The main difference is that risk redistribution can be applied to existing portfolios.

Framework

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. Consider also $0 < n \in \mathbb{N}$ insurers which will respectively pay the random indemnification $y_j \geq 0$, $j = 1, 2, .., n$, within the time period $[0, T]$. Let us suppose that $\{y_j; j = 1, 2, .., n\} \subset L^2(\mathbb{P})$, $L^2(\mathbb{P})$ being the Banach space of random variables with finite expectation and variance endowed with the usual norm $\|.\|_2$. Without loss of generality,

$$\mathbb{E}(y_j) > 0, \tag{1}$$

$j = 1, .., n$, will hold, $\mathbb{E}(.)$ denoting "mathematical expectation". These insurance companies are interested in a risk-sharing contract, and $y_{i,j}$, $i, j = \mathbf{1}, .., n$, will represent the indemnity ceded by the $i-th$ company to the $j-th$ one. In particular, $y_{j,j}$, $j = 1, .., n$, will be the indemnity retained by the $j-th$ insurer. Such a contract will be useful if every insurer reduces both risk and expected

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indemnity, where the risk measurement criterion will be discussed in Section 2.2. Consider

$$y = \sum_{i=1}^{n} y_i. \tag{2}$$

Evidently,

$$y_i = \sum_{j=1}^{n} y_{i,j}, \tag{3}$$

$i = 1, .., n$, holds, so (2) implies that

$$y = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} y_{i,j} \right). \tag{4}$$

The expected indemnity mitigation for every insurer leads to

$$\mathbb{E}(y_j) \geq \sum_{i=1}^{n} \mathbb{E}(y_{i,j}), \tag{5}$$

$j = 1, .., n$. This inequality, along with (2) and (4), leads to

$$\mathbb{E}(y) = \sum_{j=1}^{n} \mathbb{E}(y_j) \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \mathbb{E}(y_{i,j}) \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \mathbb{E}(y_{i,j}) \right) = \mathbb{E}(y),$$

and the inequality in the chain becomes an equality. Consequently, (5) becomes the equality

$$\mathbb{E}(y_j) = \sum_{i=1}^{n} \mathbb{E}(y_{i,j}), \tag{6}$$

$j = 1, .., n$, and the strict mitigation of the expected indemnification is unfeasible. In other words, the risk-sharing contract will have to guarantee a risk reduction under an identical expected indemnity for every insurer.

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Definition 1 *Consider a positive* $m \in \mathbb{N}$ *and the real-valued random variables* $U_1, U_2, .., U_m$. *They are said to be co-monotone if their joint distribution is given by the Fréchet-Hoeffding copula,*

$$c(u_1, u_2, .., u_m) = Min\{u_j;\ j = 1, .., m\}$$

for $0 \leq u_j \leq 1,\ j = 1, .., m.$ □

Definition 2 *Consider a positive* $m \in \mathbb{N}$ *and the square matrix of real-valued random variables* $(U_{i,j})_{i,j=1}^{m}$. *This matrix is said to be of co-monotone rows if* $U_{i,1}, U_{i,2}, .., U_{i,m}, \sum_{j=1}^{m} U_{i,j}$ *are co-monotone for* $i = 1, .., m.$ □

Assumption I. *Henceforth let us impose the matrix* $(y_{i,j})_{i,j=1}^{n}$ *to have co-monotone rows.* □

Framework

Proposition 3 *Fix $j = 1, 2, .., n$ and consider the functional $L^{\infty}(\mathbb{L}) \ni x \to J_j(x) \in L^2(\mathbb{P})$ given by*

$$J_j(x)(\omega) := \int_0^{y_j(\omega)} x(s)\, ds \qquad (7)$$

for $\omega \in \Omega$. Then, J_j is well-defined, linear and continuous for both the norm topologies and the weak$^*-$topologies. The adjoint $J_j^* : L^2(\mathbb{P}) \to L^1(\mathbb{L})$ is well-defined, linear, continuous for the norm and the weak$-$topologies, and it is given by

$$J_j^*(z)(s) = \int_{y_j \geq s} z(\omega)\, \mathbb{P}(d\omega)$$

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Let us take $y_{i,j} = J_i(x_{i,j})$, $i, j = 1, .., n$. Though it is an abuse of language, let us denote by 1 the obvious constant function of $L^\infty(\mathbb{L})$.[3] Then,

$$J_i(1) = y_i, \tag{9}$$

$i = 1, .., n$, is an evident consequence of (7), so (3) will hold if $\sum_{j=1}^n x_{i,j} = 1$ also holds, $i = 1, 2, .., n$. Thus, bearing in mind (6), the set $\mathcal{S} \subset (L^\infty(\mathbb{L}))^{n^2}$ of feasible risk-sharing contracts will be characterized by

$$\begin{cases} x = (x_{i,j})_{i,j=1}^n \in \mathcal{S} \subset (L^\infty(\mathbb{L}))^{n^2} \Longleftrightarrow \\ \Longleftrightarrow \begin{cases} 0 \leq x_{i,j}, \ i, j = 1, 2, .., n \\ \sum_{j=1}^n x_{i,j} = 1, \ i = 1, 2, .., n \\ \sum_{i=1}^n \mathbb{E}(J_i(x_{i,j})) = \mathbb{E}(y_j), \ j = 1, 2, .., n, \end{cases} \end{cases} \tag{10}$$

and every $x = (x_{i,j})_{i,j=1}^n \in \mathcal{S}$ will be related to the retained/ceded risks

$$y_{i,j} = J_i(x_{i,j}), \quad i, j = 1, .., n. \tag{11}$$

Obviously, (10) and (11) imply that Assumption I holds, that is, the rows of $(y_{i,j})_{i,j=1}^n$ are co-monotone.

Framework

Consider the continuous and sub-linear function $\rho_j : L^2(\mathbb{P}) \to \mathbb{R}$, $j = 1,..,n$, used by the $j-th$ insurer in order to measure and manage its risk. Let us deal with the optimization problem (see (10))

$$Min \left\{ \left(\rho_j \left(-\sum_{i=1}^{n} J_i(x_{i.j}) \right) \right)_{j=1}^{n} ; \; (x_{i.j})_{i,j=1}^{n} \in \mathcal{S} \right\}, \tag{14}$$

in order to look for the Pareto-optimal (or Pareto-efficient) risk-sharing contracts.

Remark 7 *Since $\mathcal{S}$ is a convex set (it is given by linear expressions, see (10)) and every ρ_j is a convex (sub-linear) function, every Pareto-optimum of (14) can be obtained by solving a scalar problem*

$$Min \left\{ \sum_{j=1}^{n} \alpha_j \rho_j \left(-\sum_{i=1}^{n} J_i(x_{i.j}) \right) ; \; (x_{i.j})_{i,j=1}^{n} \in \mathcal{S} \right\}, \tag{15}$$

where $\alpha_1,..,\alpha_n \geq 0$ and $\alpha_1+..+\alpha_n = 1$ (Nakayama, et al.,1985). Conversely, if in addition $\alpha_j > 0$, $j = 1,..,n$, then every solution of (15) is Pareto-efficient in (14). Though (15) draws an analogy with the classical Monge-Kantorovich mass-transfer problem (Anderson and Nash, 1987, or Jiménez-Guerra and Rodríguez-Salinas, 1996, for a very general approach), the presence of J_i, $i = 1,.,n$, is a significant modification. Accordingly, the properties of the mass-transfer problem cannot be used. □

Risk measures

A function $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be sub-linear if it is sub-additive $(\rho(u_1 + u_2) \le \rho(u_1) + \rho(u_2))$ and positively homogeneous $(\rho(\lambda u) = \lambda \rho(u)$ for $\lambda \ge 0)$. A function $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is continuous and sub-linear if and only if

$$\partial_\rho := \left\{ z \in L^2(\mathbb{P}); -\mathbb{E}(uz) \le \rho(u), \forall u \in L^2(\mathbb{P}) \right\} \tag{12}$$

is a weakly-compact subset of $L^2(\mathbb{P})$ and

$$\rho(u) = Max \ \{-\mathbb{E}(uz); \ z \in \partial_\rho\} \tag{13}$$

holds for every $u \in L^2(\mathbb{P})$. As a consequence of (12) and (13), ρ is weakly lower semi-continuous.

A function $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be decreasing if $\rho(u_1) \le \rho(u_2)$ holds when $u_1 \ge u_2$. If $E_\rho \ge 0$, then a function $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be $E_\rho-$translation invariant if $\rho(u + \lambda) = \rho(u) - E_\rho \lambda$ for $\lambda \in \mathbb{R}$, and ρ is $E_\rho-$mean dominating if $\rho(u) \ge -E_\rho \mathbb{E}(u)$. A function $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be a coherent risk measure (Artzner *et al.*, 1999) if it is continuous, sub-linear, $1-$translation invariant and decreasing. A function $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be an expectation bounded risk measure (Rockafellar *et al.*, 2006) if it is continuous, sub-linear, $1-$translation invariant and $1-$mean dominating.

Risk measures

There are many interesting examples satisfying the conditions above. For instance, the absolute deviation ($\rho(u) = \mathbb{E}(|u - \mathbb{E}(u)|)$), the standard deviation ($\rho(u) = \|u - \mathbb{E}(u)\|_2$) and the downside standard semi-deviation ($\rho(u) = \left\|(\mathbb{E}(u) - u)^+\right\|_2$) are continuos, sub-linear, $0-$translation invariant and $0-$mean dominating. Examples of coherent and expectation bounded risk measures are the $CV@R$, the $RCV@R$, the $WCV@R$ and many versions of the $WV@R$. Other usual examples such as the entropic risk measure (Kupper and Schachermayer, 2011), which is not sub-additive, the entropic $V@R$ ($EV@R$, Ahmadi-Javid, 2012), which cannot be extended to the whole $L^2(\mathbb{P})$, or some risk measures defined on Orlicz spaces (Cheridito and Tianhui, 2009), do not satisfy all the imposed properties.

Proposition 4 *Suppose that $\rho : L^2(\mathbb{P}) \to \mathbb{R}$ is sub-linear and continuous.*
a) ρ is decreasing if and only if $z \geq 0$ for every $z \in \partial_\rho$.
Suppose that $E_\rho \geq 0$.
b) ρ is $E_\rho-$translation invariant if and only if $\mathbb{E}(z) = E_\rho$ for every $z \in \partial_\rho$.
c) ρ is $E_\rho-$mean dominating if and only if $E_\rho \in \partial_\rho$. □

Optimality conditions

Proposition 8 *Problem (15) is bounded and solvable (it attains its optimal value).*

Theorem 9 *a) Problem*

$$\begin{cases} Max\ -\left(\sum_{i=1}^{n}\langle 1,\Lambda_i\rangle + \sum_{j=1}^{n}\lambda_j \mathbb{E}\left(y_j\right)\right) \\ \Lambda_i + J_i^*\left(\alpha_j z_j + \lambda_j\right) \geq 0,\ i,j=1,...,n \\ z_j \in \partial_{\rho_j},\ \Lambda_i \in L^1\left(\mathbb{L}\right),\ \lambda_j \in \mathbb{R},\ i,j=1,..,n \end{cases} \tag{17}$$

is a dual of (15), $\left((z_j)_{j=1}^n, (\Lambda_i)_{i=1}^n, (\lambda_j)_{j=1}^n\right)$ *being the decision variable. Furthermore, it is bounded and solvable, and its optimal value equals the optimal value of (15).*

b) If $\left((\tilde{z}_j)_{j=1}^m, \left(\tilde{\Lambda}_i\right)_{i=1}^n, \left(\tilde{\lambda}_j\right)_{j=1}^n\right)$ *solves (17) then*

$$\tilde{\Lambda}_i = Max\left\{-J_i^*\left(\alpha_j \tilde{z}_j + \tilde{\lambda}_j\right),\ j=1,..,n\right\}, \tag{18}$$

$i=1,...,n.$

c) If $(\tilde{x}_{i,j})_{i,j=1}^n$ *is (15)-feasible and* $\left((\tilde{z}_j)_{j=1}^n, \left(\tilde{\Lambda}_i\right)_{i=1}^n, \left(\tilde{\lambda}_j\right)_{j=1}^n\right)$ *is (17)-feasible, then they solve the corresponding problem if and only if*

$$\begin{cases} \begin{cases} \alpha_j \mathbb{E}\left(z_j \sum_{i=1}^{n} J_i\left(\tilde{x}_{i.j}\right)\right) \leq \\ \alpha_j \mathbb{E}\left(\tilde{z}_j \sum_{i=1}^{n} J_i\left(\tilde{x}_{i.j}\right)\right), \end{cases} \quad \forall z_j \in \partial_{\rho_j},\ j=1,..,n \\ \tilde{x}_{i,j}\left(\tilde{\Lambda}_i + J_i^*\left(\alpha_j \tilde{z}_j + \tilde{\lambda}_j\right)\right) = 0, \quad i,j=1,..,n. \end{cases} \tag{19}$$

Optimality conditions

d) If $(\tilde{x}_{i,j})_{i,j=1}^{n}$ *solves (15) and* $\left((\tilde{z}_j)_{j=1}^{m}, \left(\tilde{\Lambda}_i\right)_{i=1}^{n}, \left(\tilde{\lambda}_j\right)_{j=1}^{n}\right)$ *solves (17), then* $(\tilde{x}_{i,j})_{i,j=1}^{n}$ *solves the linear problem*

$$\begin{cases} Min \; \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_j \mathbb{E}\left(\tilde{z}_j J_i\left(x_{i.j}\right)\right) \\ (x_{i.j})_{i,j=1}^{n} \in \mathcal{S}, \end{cases} \qquad (20)$$

and the optimal objectives of (15) and (20) coincide.

Systemic risk

Consider the global indemnification $y \in L^2(\mathbb{P})$ of (2) and a convex cone V such that

$$y \in V \subset L^2_+(\mathbb{P}) := \left\{v \in L^2(\mathbb{P}); 0 \leq v\right\}. \quad (23)$$

Important particular cases are $V = L^2_+(\mathbb{P})$, $V = V_y$, where

$$V_y := \left\{v \in L^2; 0 \leq v \leq \eta y \ for\ some\ \eta \in \mathbb{R}, \eta \geq 0\right\}$$

and $V = V_{(J_1,...,J_n)}$, where

$$V_{(J_1,...,J_n)} := \left\{v = \sum_{i=1}^{n} J_i(x_i); x_i \in L^\infty(\mathbb{L}),\ x_i \geq 0,\ i = 1,..,n\right\}. \quad (24)$$

Throughout Section 4.2 let us suppose that $E_\rho \geq 0$, and consider the sub-linear, continuous, $E_\rho-$translation invariant and $E_\rho-$mean dominating function $\rho : L^2(\mathbb{P}) \rightarrow \mathbb{R}$, and the linear, continuous and (non necessarily strictly) increasing function $\Psi : L^2(\mathbb{P}) \rightarrow \mathbb{R}$. According to the Riesz theorem (Zeidler, 1995), there exists a unique $z_\Psi \in L^2(\mathbb{P})$, $z_\Psi \geq 0$, such that $\Psi(u) = \mathbb{E}(z_\Psi u)$ for every $u \in L^2(\mathbb{P})$.

Systemic risk

Consider an insurer which deals with ρ to manage its risk and with Ψ to price its insurance policies. If the insurer could select the total indemnification $v \in V$ to be paid and the expected profit $M > 0$ to be reached, it would solve the optimization problem

$$Min \ \{\rho(-v);\ \mathbb{E}(v(z_\Psi - 1)) \geq M,\ v \in V\}. \qquad (25)$$

Suppose that

$$\mathbb{E}(y(z_\Psi - 1)) > 0. \qquad (26)$$

Proceeding as in (16), the Lagrangian of (25) becomes

$$\mathcal{L}(v, z, \lambda) = \mathbb{E}(v(\lambda + z - \lambda z_\Psi)) + M\lambda$$

for $(v, z, \lambda) \in V \times \partial_\rho \times \mathbb{R}$ with $\lambda \geq 0$, and the dual of (25) becomes

$$\begin{cases} Max \ M\lambda \\ \lambda + z - \lambda z_\Psi \in V^* \\ (z, \lambda) \in \partial_\rho \times \mathbb{R},\ \lambda \geq 0, \end{cases} \qquad (27)$$

Systemic risk

where $V^* = \left\{u \in L^2(\mathbb{P}) ; \mathbb{E}(uv) \geq 0 \ \forall v \in V\right\}$ is the usual dual cone of V. Furthermore, there is no duality gap between (25) and (27) because (26) implies the fulfillment of the Slater qualification (Luenberger, 1969). In particular, the necessary and sufficient optimality conditions for (25) and (27) are

$$\begin{cases} \mathbb{E}(\tilde{z}\tilde{v}) \geq \mathbb{E}(z\tilde{v}), \ \forall z \in \partial_\rho \\ \mathbb{E}\left(\tilde{v}\left(\tilde{\lambda} + \tilde{z} - \tilde{\lambda} z_\Psi\right)\right) = 0 \\ \tilde{\lambda}\left(\mathbb{E}\left(\tilde{v}\left(z_\Psi - 1\right)\right) - M\right) = 0. \end{cases} \quad (28)$$

Notice that (25) is bounded because (23) and the properties of ρ imply that $\rho(-v) \geq E_\rho \mathbb{E}(v) \geq 0$ holds if v is (25)-feasible. Thus, Theorem 10 below is presented without proof because the non proved parts are analogous to those of Theorem 9.

Systemic risk

Theorem 10 *a)* *(25) and (27) are feasible and bounded. Moreover, (27) is solvable and both optimal values coincide.*

b) If $\tilde{v}$ *and* $\left(\tilde{z}, \tilde{\lambda}\right)$ *are (25)-feasible and (27)-feasible respectively, then they solve the corresponding problem if and only if (28) holds.* □

Corollary 11 *a) The (27)-feasible set and the solutions of this problem do not depend on* M*. Moreover, if there are more than one solution, then all of them have the same component* $\tilde{\lambda}$*, and only the first component* $\tilde{z}$ *may become different.*

b) (25) is solvable (i.e., attains its optimal value) if and only if so does (25) for $M = 1$*, in which case* $\tilde{v}$ *solves (25) for* $M = 1$ *if and only if* $M\tilde{v}$ *solves (25).*

c) If $\left(\tilde{z}, \tilde{\lambda}\right)$ *solves (27) (and therefore* $\tilde{\lambda}$ *is the optimal value of (25) and (27) for* $M = 1$*), then* $\tilde{\lambda} \leq \rho(-y) / \mathbb{E}((z_\Psi - 1) y)$ *(see (26)).*

d) If $z_\Psi \leq 1 + \xi$ *with* $\xi > 0$ *and* $\left(\tilde{z}, \tilde{\lambda}\right)$ *solves (27), then* $\tilde{\lambda} \geq E_\rho / \xi$*.*

Definition 12 *a) The component* $\tilde{\lambda}$ *of the solution of (27) will be said to be the systemic risk of* (y, V, ρ, z_Ψ)*.*

b) If $\tilde{v}$ *solves (25) for* $M = 1$*, then* $\tilde{v}$ *will be said to be a benchmark policy of* (y, V, ρ, z_Ψ)*.*[8] □

Systemic risk

Remark 13 *a*) *Under an upper bound for* z_Ψ*, Corollary* 11*d yields a lower bound for the systemic risk, which becomes strictly positive if* $E_\rho > 0$ *(CV@R, WV@R, WCV@R, RCV@R, etc.).*

b) *If* $E_\rho = 0$ *(absolute deviation, standard deviation, downside standard semi-deviation, etc.), then the systemic risk* $\tilde{\lambda}$ *might vanish. For instance, if* $V = L^2_+(\mathbb{P})$ *and* $z_\Psi = 1 + \xi$ *with* ξ *strictly positive, the dual constraint implies that* $\lambda + z - \lambda(1+\xi) \geq 0$*, i.e.,* $z \geq \lambda\xi$*. Taking expectations, and bearing in mind Proposition* 4*b*, $0 \geq \lambda\xi$*, and therefore* $0 = \lambda\xi$ *because* $\lambda \geq 0$*. Since* $\xi > 0$*, one has* $\lambda = 0$ *for every (27)-feasible* (z, λ)*.*

c) *Notice that Corollary* 11*c yields an upper bound of the systemic risk.*

d) *Corollary* 11*a shows that the non formal expression*

$$\text{“}Risk = \tilde{\lambda} \times Guaranteed - Expected\_\operatorname{Pr}ofit\text{”}$$

holds, $\tilde{\lambda}$ *being the systemic risk of* (y, V, ρ, z_Ψ) *and under an efficient selection by the insurer. An “inefficient insurer” will face a risk level higher than the product of the systemic risk and its guaranteed expected profit.*

e) *Suppose that a benchmark* $\tilde{v}$ *of* (y, V, ρ, z_Ψ) *exists. If* $\tilde{\lambda} > 0$*, then the third condition in (28) implies that* $\mathbb{E}(\tilde{v}(z_\Psi - 1)) = 1$*. In other words, under optimality, the guaranteed expected profit equals the expected profit.* □

Background risk

Consider again the same notation as in Section 3. Proposition 8 guarantees that (15) is solvable. In particular, for $k = 1, .., n$, one can consider $(\alpha_j)_{j=1}^n$ such that $\alpha_k = 1$ and $\alpha_j = 0$ if $j \neq k$, a corresponding solution $\left(\tilde{x}_{i,j}^{(k}\right)_{i,j=1}^n$ of (15), and the optimal objective value $I_k = \rho_k \left(-\sum_{i=1}^n J_i \left(\tilde{x}_{i,k}^{(k} \right) \right)$.

Definition 14 *Vector*

$$(I_j)_{j=1}^n := \left(\rho_j \left(-\sum_{i=1}^n J_i \left(\tilde{x}_{i,j}^{(j} \right) \right) \right)_{j=1}^n \in \mathbb{R}^n \tag{29}$$

will be said to be the ideal point of (14). □

Proposition 15 *Suppose that $E_{\rho_j} \geq 0$ and every ρ_j is both $E_{\rho_j}-$mean dominating and $E_{\rho_j}-$translation invariant, $j = 1, ..., n$. Consider the convex cone (24). Consider $\xi > 0$, $z_\Psi = 1 + \xi$ and $M_j = \xi \mathbb{E}(y_j)$, $j = 1, .., n$. Consider finally the ideal point $(I_j)_{j=1}^n$ and the systemic risks $\tilde{\lambda}_j$ of $\left(y, V_{(J_1,...,J_n)}, \rho_j, z_\Psi\right)$, $j = 1, ..., n$. Then, $I_j \geq M_j \tilde{\lambda}_j$, $j = 1, .., n$.*

Background risk

Proposition 17 *Consider a Pareto-solution* $(\tilde{x}_{i,j})_{i,j=1}^{n}$ *of (14) (whose existence is guaranteed by Proposition 8). The* $j-th$ *objective* $\rho_j\left(-\sum_{i=1}^{n} J_i(\tilde{x}_{i.j})\right)$ *and the ideal point satisfy* $\rho_j\left(-\sum_{i=1}^{n} J_i(\tilde{x}_{i.j})\right) \geq I_j$, $j = 1,..,n$. *In particular, if* $\xi > 0$, $z_\Psi = 1 + \xi$, $M_j = \xi \mathbb{E}(y_j)$ *and* $\tilde{\lambda}_j$ *is the systemic risk of* $\left(y, V_{(J_1,..,J_n)}, \rho_j, z_\Psi\right)$, *then*

$$\tilde{\lambda}_j \leq \frac{1}{M_j}\rho_j\left(-\sum_{i=1}^{n} J_i(\tilde{x}_{i.j})\right),$$

$j = 1,..,n.$

Background risk

Definition 18 *Under the conditions of the latter proposition,*

$$\rho_j\left(-\sum_{i=1}^{n} J_i\left(\tilde{x}_{i.j}\right)\right)$$

will be called the background risk of the $j-th$ insurer under the optimal risk-sharing contract $\left(\tilde{x}_{i,j}\right)_{i,j=1}^{n}$. The difference

$$\rho_j\left(-\sum_{i=1}^{n} J_i\left(\tilde{x}_{i.j}\right)\right) - I_j \geq 0$$

may be interpreted as he "gap" between the background risk and the ideal one, which could be achieved if the $j-th$ insurer were the unique decision maker in the risk-sharing plan. Similarly, the difference

$$\frac{1}{M_j}\rho_j\left(-\sum_{i=1}^{n} J_i\left(\tilde{x}_{i.j}\right)\right) - \tilde{\lambda}_j \geq 0$$

may be interpreted as the "gap" between the background risk per dollar of expected profit and the systemic ("invincible") risk per dollar of expected profit, which could be achieved if the $j-th$ insurer could decide the set of policies to sell. □

Coalitions, bargaining set and core

every optimal sharing contract $(x_{i,j})_{i,j=1}^{n} \in \mathcal{S}$ also implies the collaboration of all the involved insurers, it is natural to deal with the key notions in cooperative games, that is, the concepts of coalition, core and bargaining set (Aumann and Maschler, 1964, or Maschler, 1976). Consider the set $A = \{1, 2, .., n\}$ and the set 2^A composed of the subsets of A. The $j-th$ insurer will be identified with the elements $j \in A$ or $\{j\} \in 2^A$. Every $B \in 2^A \setminus \{\varnothing\}$ will represent a coalition containing those insurers belonging to B. In particular, A will be called the grand coalition. If $B \in 2^A \setminus \{\varnothing\}$ contains m elements $(0 < m \leq n)$, $W_B \subset \mathbb{R}^n$ will be the compact set

$$W_B := \left\{ \alpha = (\alpha_j)_{j \in B};\ \alpha_j \geq 0 \ for \ j \in B,\ \sum_{j \in B} \alpha_j = 1 \right\} \subset \mathbb{R}^m. \qquad (32)$$

Similarly, $\mathcal{S}^{(B}$ will denote the subset of $(L^\infty(\mathbb{L}))^{m^2}$ which is analogous to $\mathcal{S}$ (see (10)). Obviously, as $\mathcal{S}$, $\mathcal{S}^{(B}$ is non void (Proposition 5) and $weakly^*-$compact. Henceforth we will denote

$$\mathcal{P}(A) := \left\{ \left(B, x = (x_{i,j})_{i,j \in B} \right);\ B \in 2^A \setminus \{\varnothing\} \ and \ x \in \mathcal{S}^{(B} \right\},$$

and

$$\rho_B(x) := \left(\rho_j \left(-\sum_{i \in B} J_i(x_{i.j}) \right) \right)_{j \in B} \in \mathbb{R}^m$$

for every $x = (x_{i,j})_{i,j \in B} \in (L^\infty(\mathbb{L}))^{m^2}$.

Coalitions, bargaining set and core

Lemma 19 *If $x \in \mathcal{S}$, then there exists a Pareto-solution $\tilde{x}$ of (14) such that $\rho_A(\tilde{x}) \leq \rho_A(x)$.* □

Lemma 19 allows us to slightly simplify the formal definitions of core and bargaining set. Let us adapt the definitions given in Hervés-Estévez and Moreno-García (2018).

Definition 20 *a) Given a Pareto-efficient (or Pareto-optimal) solution $\tilde{x} = (\tilde{x}_{i,j})_{i,j=1}^{n}$ of (14) and $\left(B, x = (x_{i,j})_{i,j=1}^{n}\right) \in \mathcal{P}(A)$, it will be said that (B, x) blocks (or objects) $\tilde{x}$ if*

$$\rho_j\left(-\sum_{i \in B} J_i(x_{i.j})\right) \leq \rho_j\left(-\sum_{i=1}^{n} J_i(\tilde{x}_{i.j})\right)$$

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holds for every $j \in B$, *and at least one inequality is strict.*

b) Suppose that $\tilde{x} = (\tilde{x}_{i,j})_{i,j=1}^{n}$ *is a Pareto-solution of (14) objected by* $\left(B, x^{(B} = \left(x_{i,j}^{(B}\right)_{i,j=1}^{n}\right) \in \mathcal{P}(A)$. *Consider* $\left(C, x^{(C} = \left(x_{i,j}^{(C}\right)_{i,j=1}^{n}\right) \in \mathcal{P}(A)$ *such that* $C \cap B \neq \varnothing$ *and* $C \setminus B \neq \varnothing$. *It will be said that* $\left(C, x^{(C}\right)$ *counter-blocks (or counter-objects)* $\left(B, x^{(B}\right)$ *if*

$$\rho_j\left(-\sum_{i\in C} J_i\left(x_{i.j}^{(C}\right)\right) \le \rho_j\left(-\sum_{i\in B} J_i\left(x_{i.j}^{(B}\right)\right) \tag{33}$$

holds for every $j \in C \cap B$ *and*

$$\rho_j\left(-\sum_{i\in C} J_i\left(x_{i.j}^{(C}\right)\right) \le \rho_j\left(-\sum_{i=1}^{n} J_i\left(\tilde{x}_{i.j}\right)\right) \tag{34}$$

holds for every $j \in C \setminus B$, *with at least a strict inequality in (33) or (34).*

c) The core will be composed of those Pareto-solutions of (14) which are not blocked by any $(B, x) \in \mathcal{P}(A)$.

d) The bargaining set will be composed by those Pareto solutions x *of (14) such that either,* x *is in the core, or for every* $\left(B, x^{(B}\right) \in \mathcal{P}(A)$ *blocking* x *there exists* $\left(C, x^{(C}\right) \in \mathcal{P}(A)$ *counter-blocking* $\left(B, x^{(B}\right)$. □

Conclusion

- Highly correlated risks are becoming more and more usual for insurance companies.
- Practical solutions are related to financial markets by means of ILS (insurance linked securities) and/or parametric insurance (actuarial índices)
- An alternative approaches are related to risk-sharing problems.
- Our novelty is the co-monotonic risk-redistribution.
- With respect to the risk-sharing approach, the risk-redistribution one allows us to deal with existing portfolios.

Conclusion

- This approach leads to a convex multióbjective optimization problem whose dual and Pareto solution have been properly characterized. Actually, the problem may be solved by means of the given conditions.
- Nevertheless, this new approach generates other problems which are absent when dealing with risk-sharing. Indeed, the usual core and bargaining set of cooperative games have to be studied
- Partial conclusions may be reached by dealing with the systemic and the background risk, but further theoretical studies are needed.

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Thanks for your attention

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